

Cubic convergence of parameter-controlled Newton-secant method for multiple zeros

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ABSTRACT

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ have a multiple zero α with integer multiplicity $m \geq 1$ and be analytic in a sufficiently small neighborhood of α . For parameter-controlled Newton-secant method defined by

$$x_{n+1} = x_n - \frac{\lambda f(x_n)^2}{f'(x_n) \cdot \{f(x_n) - f(x_n - \mu f(x_n)/f'(x_n))\}}, \quad n = 0, 1, 2, \dots,$$

we investigate the maximal order of convergence and the theoretical asymptotic error constant by seeking the relationship between parameters λ and μ . For various test functions, the numerical method has shown a satisfactory result with high-precision Mathematica programming.

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1. Introduction

Newton-secant method or leap-frogging Newton's method was introduced by many researchers including Traub [1] and Kasturiarachi [2]. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ have a multiple zero α with integer multiplicity $m \geq 1$ and be analytic [3,4] in a sufficiently small neighborhood of α . In this paper, we consider its extended version below:

$$x_{n+1} = x_n - \frac{\lambda f(x_n)^2}{f'(x_n) \cdot \{f(x_n) - f(z_n)\}}, \quad z_n = x_n - \mu \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where λ and μ are parameters to be chosen for maximal order of convergence. We are interested in seeking the relationship between $\lambda \neq 0$ and $\mu \neq 0$ as well as expressing the theoretical asymptotic error constant [5–8,1] for maximal order of convergence. Many Newton-like numerical methods of order 3 have been studied in [9,7,10–12,1]. Especially, the case where $\lambda = 1$, $\mu = 1$ and $m = 1$ has been investigated in [7,2,1]. Our proposed numerical scheme (1.1) has the advantage that it is free of second derivatives, unlike other third-order numerical schemes for multiple zeros including Halley-like method

$$x_{n+1} = x_n - \frac{2}{(1 + 1/m) \frac{f'(x_n)}{f(x_n)} - \frac{f''(x)}{f'(x_n)}}.$$

We rewrite $f(x) = 0$ in the form $x = g(x)$ with $g : \mathbb{C} \rightarrow \mathbb{C}$ being analytic in a small neighborhood of α . Suppose that the iterative function [5,6,1] $g(x)$ is defined by

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$$g(x) = \begin{cases} x - \frac{\lambda f(x)^2}{f'(x) \cdot \{f(x) - f(z)\}}, & z = x - \mu h(x), \quad \text{if } x \neq \alpha \\ \alpha, & \text{if } x = \alpha, \end{cases} \quad (1.2)$$

where

$$h(x) = \begin{cases} f(x)/f'(x), & \text{if } x \neq \alpha \\ \lim_{x \rightarrow \alpha} f(x)/f'(x), & \text{if } x = \alpha. \end{cases} \quad (1.3)$$

Hence our proposed numerical method (1.1) is given by

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \dots, \quad (1.4)$$

and is called extended leap-frogging Newton's method. By a similar analysis conducted in [7] with $f : \mathbb{C} \rightarrow \mathbb{C}$, it can be shown that (1.4) converges to α , if x_0 is chosen in a sufficiently small neighborhood of α . As expected with $\lambda = m = 1$, it becomes Newton-secant method.

The following lemma describing some local properties of $h(x)$ can be verified by repeated use of L'Hospital's rule [4] and Leibniz rule [4] for differentiation.

Lemma 1.1. Let $\theta_k = \frac{f^{(m+k)}(\alpha)}{f^{(m)}(\alpha)}$ for $k \in \mathbb{N}$ with f, α and m described in Section 1. Then the function $h(x)$ defined in (1.3) and its derivatives up to order 3 evaluated at α satisfy the following relations:

- (i) $h(\alpha) = 0$.
- (ii) $h'(\alpha) = \frac{1}{m}$.
- (iii) $h''(\alpha) = -\frac{2\theta_1}{m^2(m+1)}$.
- (iv) $h'''(\alpha) = \frac{6}{m^2(m+1)} \left\{ \frac{1}{m} \theta_1^2 - \frac{2}{m+2} \theta_2 \right\}$.

To investigate some local properties of $g(x)$ in a small neighborhood of α , we rewrite (1.1) as

$$(g - x) \cdot f'(x) \{f(x) - f(z)\} = -\lambda \cdot f(x)^2 \quad (1.5)$$

where $f = f(x)$, $f' = f'(x)$, $z = x - \mu h(x)$ are used for convenience and the symbol $'$ denotes the derivative with respect to x . Using Eq. (1.5), our aim is to establish some relationship between λ , m , $g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$ for maximum order of convergence. The following lemma is useful to calculate $g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$. By direct computation with the aid of Lemma 1.1, it can be proved without difficulty and hence its proof is omitted here.

Lemma 1.2. Let $z(x) = x - \mu h(x) \in \mathbb{C}$ with $\mu \neq 0$ be as described in the beginning of Section 1. Let $z'(x) = 1 - \frac{\mu}{m} = t \neq 1$ with $m \in \mathbb{N}$. Let θ_k for $k \in \mathbb{N}$ be as described in Lemma 1.1. Let $F_k = \frac{d^k}{dx^k} f(z)|_{x=\alpha}$ for $k = 0, 1, 2, \dots$. Then the following hold:

- (i) $F_k = 0$ for $0 \leq k \leq m-1$.
- (ii) $F_m = f^{(m)}(\alpha) t^m$ for $k = m$.
- (iii) $F_{m+1} = f^{(m)}(\alpha) \cdot \theta_1 \cdot t^{m-1} (1-t+t^2)$ with $t^0 \equiv 1, \forall t \in \mathbb{C}$.
- (iv) $F_{m+2} = f^{(m)}(\alpha) \cdot t^{m-2} \cdot \left\{ \theta_2 t^4 + \frac{m+2}{m} \theta_1^2 t^2 (1-t) + \frac{m+2}{2} (1-t) H_0 \right\}$ with $H_0 = \left\{ -2t + \frac{m-1}{m+1} (1-t) \right\} \frac{\theta_2^2}{m} + \frac{4t}{m+2} \theta_2, t^0 \equiv 1, \forall t \in \mathbb{C}$.

2. Convergence analysis

In this section, we investigate the asymptotic error constant η in terms of $t = 1 - \mu/m \neq 1$ with parameters λ and μ described in (1.1) to obtain the maximal order of convergence p .

Differentiating $2m$ times both sides of Eq. (1.5) with respect to x and substituting $x = \alpha$, we obtain the following with the aid of Leibniz rule for differentiation:

$$\sum_{r=0}^{2m} \binom{2m}{r} (g-x)^{(2m-r)} \cdot [f' \cdot \{f-f(z)\}]^{(r)}|_{x=\alpha} = -\lambda \sum_{r=0}^{2m} \binom{2m}{r} f^{(r)} \cdot f^{(2m-r)}|_{x=\alpha}. \quad (2.1)$$

Using $f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0, f^{(m)}(\alpha) \neq 0$ and Lemma 1.2, we favorably find that $[f' \cdot \{f-f(z)\}]^{(r)}|_{x=\alpha} \neq 0$ on the left-hand side of (2.1) only when $r = 2m-1$. By Lemma 1.2, when $r = 2m-1$, we have

$$[f' \cdot \{f-f(z)\}]^{(2m-1)}|_{x=\alpha} = \binom{2m-1}{m-1} f^{(m)}(\alpha)^2 (1-t^m). \quad (2.2)$$

Using (2.2) in (2.1) yields

$$2m \cdot (g' - 1) \cdot \binom{2m-1}{m-1} f^{(m)}(\alpha)^2 (1-t^m) = -\lambda \binom{2m}{m} \cdot f^{(m)}(\alpha)^2.$$

Next we determine the value λ such that $g'(\alpha) = 0$ and find that for all $m \geq 1$

$$\lambda = m(1 - t^m). \quad (2.3)$$

With such a λ , we further differentiate $2m + 1$ times both sides of Eq. (1.5) with respect to x and substitute $x = \alpha$ as follows:

$$\sum_{r=0}^{2m+1} \binom{2m+1}{r} (g-x) \big|_{x=\alpha}^{(2m+1-r)} [f' \cdot \{f-f(z)\}]_{x=\alpha}^{(r)} = -m(1-t^m) \sum_{r=0}^{2m+1} \binom{2m+1}{r} f^{(r)} \cdot f^{(2m+1-r)} \big|_{x=\alpha}. \quad (2.4)$$

When $r = 2m - 1$ and $r = 2m$, the left-hand side of (2.4) can have nonzero terms.

(i) When $r = 2m - 1$, using (2.2), we have the following.

$$[f' \cdot \{f-f(z)\}]_{x=\alpha}^{(2m-1)} = \binom{2m-1}{m-1} f^{(m)}(\alpha)^2 (1-t^m). \quad (2.5)$$

(ii) When $r = 2m$, by Lemma 1.2, we get the following:

$$\begin{aligned} [f' \cdot \{f-f(z)\}]_{x=\alpha}^{(2m)} &= \binom{2m}{m} f^{(m+1)}(\alpha) [f^{(m)}(\alpha) - f(z)^{(m)}]_{x=\alpha} \\ &\quad + \binom{2m}{m-1} f^{(m)}(\alpha) [f^{(m+1)}(\alpha) - f^{(m+1)}(\alpha) t^{m-1} (1-t+t^2)]_{x=\alpha} \\ &= \frac{2m!}{(m+1)!m!} f^{(m+1)}(\alpha) f^{(m)}(\alpha) \Delta_0, \end{aligned} \quad (2.6)$$

where $\Delta_0 = 2m + 1 - t^{m-1}(m + t + mt^2)$. Therefore, we let $g'(\alpha) = 0$ on the left-hand side of Eq. (2.4) to obtain the following:

$$\begin{aligned} &-\frac{(2m+1)!}{(m+1)!m!} f^{(m)}(\alpha) f^{(m+1)}(\alpha) \Delta_0 + \frac{1}{2} g''(\alpha) \frac{(2m+1)!}{(m-1)!m!} f^{(m)}(\alpha)^2 (1-t^m) \\ &= -m(1-t^m) \left[\binom{2m+1}{m} + \binom{2m+1}{m+1} \right] f^{(m)}(\alpha) f^{(m+1)}(\alpha). \end{aligned}$$

Hence with θ_1 defined in Lemma 1.1, we obtain after expressing Δ_0 in terms of m and t :

$$g''(\alpha) = \frac{2\theta_1}{m(m+1)} \cdot \frac{\rho(t)}{V_{m-1}}, \quad (2.7)$$

where $\rho(t) = mt^{m-1}(t-1) + V_{m-1}$ and $V_k = 1 + t + t^2 + \dots + t^k$ for $k \in \mathbb{N} \cup \{0\}$ with $t^0 \equiv 1$ for any $t \in \mathbb{C}$. Observe that $\rho(t) = t$ when $m = 1$. We next find t such that $g''(\alpha) = 0$ as m varies to obtain possible third-order convergence. To this end, we seek roots of $\rho(t) = 0$. Since $\rho(t)$ is a polynomial in t , by the fundamental theorem of algebra [4], there exists a $t \in \mathbb{C}$ for which $\rho(t) = 0$. Indeed, such a t is an algebraic number [13] since all the coefficients are integers. Some theoretical nature of such a root t can be stated in the following theorem:

Theorem 2.1. Let ρ be defined as in (2.7). Then ρ has a unique real root $t^* \in (-1, 0]$ for any odd m , and no real root for any even m .

Proof. If $m = 1$ and $m = 2$, then direct computation assures $t^* \in (-1, 0]$ uniquely exists. In fact, $t^* = 0$ occurs only when $m = 1$. If $m = 3$, then $\rho(t) = 3t^3 - 2t^2 + t + 1$ and $\rho(-1) \cdot \rho(0) < 0$ guarantees the existence of a $t^* \in (-1, 0)$ with $\rho(t^*) = 0$. Since $\rho(-t) = -3t^3 - 2t^2 - t + 1$ has one sign change in its coefficients, ρ has at most one negative real root according to Descartes' Sign Rule [6]; consequently, t^* is unique. Observe that there exists no such value of t simultaneously satisfying $\rho(t) = V_{m-1} = 0$; if it does, then $t = 0$ or $t = 1$ contradicting $V_{m-1} = 0$. Hence (2.7) is well-defined for any t satisfying $\rho(t) = 0$.

To continue the proof for $m \geq 4$, we rewrite ρ with $t \neq 1$ as

$$\rho(t) = mt^{m-1}(t-1) + (t^m - 1)/(t-1). \quad (2.8)$$

For convenient analysis, we introduce its derivative

$$\rho'(t) = \sigma(t)/(t-1)^2 \quad (2.9)$$

along with

$$\sigma(t) = m(t-1)t^{m-2}\{mt^2 - 2(m-1)t + m-1\} + 1 - t^m, \quad (2.10)$$

$$\sigma'(t) = m^2(m+1)(t-1)t^{m-3} \cdot q(t), \quad (2.11)$$

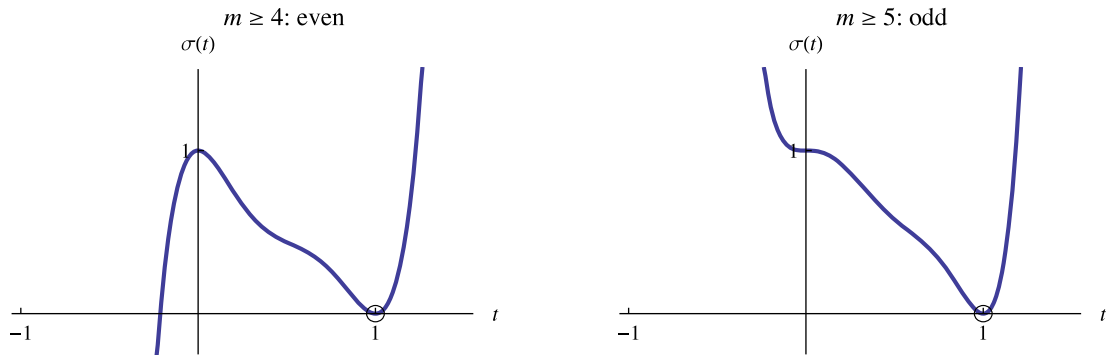


Fig. 1. Sketch of $\sigma(t)$ for an even $m \geq 4$ and an odd $m \geq 5$.

where $q(t) = \left(t - \frac{(m-1)(2m-1)}{2m(m+1)}\right)^2 + \frac{(m-1)(4m^2-13m+1)}{4m(m+1)}$. The sign variation of $\sigma'(t)$ needs to be checked only in a neighborhood of $t = 1$ and $t = 0$ since $q(t) > 0$ for all real t with $t \neq 1$. By elementary calculus, σ is shown to be a piecewise monotone function and its graph is sketched in Fig. 1. The monotonicity enables us to find that $\rho'(t) > 0$ for all $t \geq 0$ with $t \neq 1$. Consequently, $\rho(t) > \rho(0) = 1 > 0$ for all $t \geq 0$, which shows ρ has no real root. When $m \geq 5$ is odd, $\sigma(t) > 0$ for all $t < 0$ in view of its monotonicity; hence, ρ as a polynomial of an odd degree is monotone increasing for all $t < 0$ and has a unique negative root $t^* \in (-1, 0)$ by the intermediate-value theorem [14] since $\rho(-1) \cdot \rho(0) = 1 - 2m < 0$. When $m \geq 4$ is even, $\sigma(t)$ is monotone decreasing for all $t < 0$ and has a unique negative real root $t_0 \in (-1, 0)$ since $\sigma(0) \cdot \sigma(-1) = -2m(4m-3) < 0$. Indeed, t_0 is a global minimum point since $\rho'(t_0) = 0$ and ρ is a polynomial of an even degree. Direct computation with $\sigma(t_0) = 0$ shows that $\rho(t_0) = m t_0^{m-2} \left[(m+1) \left(t_0 - \frac{(1-2m)}{2(m+1)} \right)^2 + \frac{4m-5}{4(m+1)} \right] > 0$. Hence, $\rho(t) \geq \rho(t_0) > 0$ for any $t < 0$ has no negative real root, completing the entire proof. \square

Suppose we are given a $t \in \mathbb{C}$. If $\rho(t) \neq 0$ with $V_{m-1} \neq 0$, then (1.1) converges to α quadratically. If $\rho(t) = 0$, then (1.1) converges to α cubically. We are indeed interested in deriving the asymptotic error constant for cubic convergence. To this end, assuming t is a root of $\rho(t)$, we further differentiate $2m+2$ times both sides of Eq. (1.5) with respect to x and substitute $x = \alpha$ as follows:

$$\sum_{r=0}^{2m+2} \binom{2m+2}{r} (g-x) \Big|_{x=\alpha}^{(2m+2-r)} \cdot [f' \cdot \{f-f(z)\}]_{x=\alpha}^{(r)} = -m(1-t^m) \cdot \sum_{r=0}^{2m+2} \binom{2m+2}{r} f^{(r)} \cdot f^{(2m+2-r)} \Big|_{x=\alpha}. \quad (2.12)$$

The left-hand side of (2.12) possibly has nonzero values if $r = 2m-1, 2m$ and $r = 2m+1$.

(i) When $r = 2m-1$, we have from (2.2)

$$[f' \cdot \{f-f(z)\}]_{x=\alpha}^{(2m-1)} = \binom{2m-1}{m-1} f^{(m)}(\alpha)^2 (1-t^m).$$

(ii) When $r = 2m$, we have from (2.6)

$$[f' \cdot \{f-f(z)\}]_{x=\alpha}^{(2m)} = \frac{2m!}{(m+1)!m!} f^{(m)}(\alpha) f^{(m+1)}(\alpha) \Delta_0.$$

(iii) When $r = 2m+1$, from Lemma 1.2, we have

$$[f' \cdot \{f-f(z)\}]_{x=\alpha}^{(2m+1)} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} f^{(k)} \Big|_{x=\alpha} \cdot (f-f(z)) \Big|_{x=\alpha}^{(2m+1-k)} = \binom{2m+1}{m-1} f^{(m)}(\alpha)^2 W,$$

where $W = \theta_2 - T + \frac{m+2}{m} \theta_1^2 (1-t^{m-1}(1-t+t^2)) + \frac{m+2}{m} \theta_2 (1-t^m)$ with θ_1, θ_2 defined in Lemma 1.1, and $T = t^{m+2} [\theta_2 t^4 + \frac{m+2}{m} \theta_1^2 t^2 (1-t) + \frac{m+2}{m} (1-t) H_0]$ with H_0 defined in Lemma 1.2. Using $g'(\alpha) = 0$ and $g''(\alpha) = 0$ on the left-hand side of Eq. (2.12) we find:

$$\begin{aligned} & -2(m+1) \cdot \binom{2m+1}{m-1} f^{(m)}(\alpha)^2 \cdot W + \frac{(m+1)^2(m+2)}{3} g'''(\alpha) \binom{2m+1}{m-1} f^{(m)}(\alpha)^2 (1-t^m) \\ & = -m(1-t^m) \cdot f^{(m)}(\alpha)^2 \left[2 \binom{2m+2}{m} \theta_2 + \binom{2m+2}{m+1} \theta_1^2 \right]. \end{aligned}$$

From this, it follows that

$$g'''(\alpha) = \frac{6}{(m+1)(m+2)} \left(\frac{W}{1-t^m} - 2\theta_2 - \frac{m+2}{m+1} \theta_1^2 \right). \quad (2.13)$$

Table 1Values ρ , t and η for $1 \leq m \leq 5$.

m	$\rho(t)$	t	η
1	t	0	$\frac{1}{4}\theta_1^2$
2	$2t^2 - t + 1$	$\frac{1 \pm \sqrt{7}i}{4}, i = \sqrt{-1}$	$\frac{1}{12} \phi_1\theta_1^2 + \phi_2\theta_2 ^a$
3	$3t^3 - 2t^2 + t + 1$	$\frac{1}{9}(2 + 1/a - 5a) \simeq -0.4211548146942765$, where $a = \sqrt[3]{\frac{2}{-281+27\sqrt{109}}} \simeq 1.31067213007138$	$\frac{1}{20} \phi_1\theta_1^2 + \phi_2\theta_2 ^b$
4	$4t^4 - 3t^3 + t^2 + t + 1$	$\frac{1}{96}(18 + 6b \pm \sqrt{3}c) \simeq 0.710616 \pm 0.657465i$, where $b = \sqrt{\frac{1}{3}(-5 + 464d + \frac{32}{d})}$, $c = \sqrt{8(-5 - 232d - \frac{16}{d} - \frac{447}{b})}$, $d = \frac{2}{(46+6i\sqrt{5361})^{1/3}}, i = \sqrt{-1}$.	$\frac{1}{30} \phi_1\theta_1^2 + \phi_2\theta_2 ^c$
5	$5t^5 - 4t^4 + t^3 + t^2 + t + 1$	-0.5444346547467631	$\frac{1}{42} \phi_1\theta_1^2 + \phi_2\theta_2 ^d$

$$^a \phi_1 = \frac{1+3t}{3+3t}, \phi_2 = \frac{1-t+t^2+t^3}{1+t}.$$

$$^b \phi_1 = \frac{5(1+2t^2)}{122(1+t+t^2)}, \phi_2 = \frac{2+2t-4t^2+3t^3+3t^4}{3(1+t+t^2)}.$$

$$^c \phi_1 = \frac{3(2+2t-t^2+5t^3)}{20(1+t+t^2+t^3)}, \phi_2 = \frac{1+t+t^2-3t^3+2t^4+2t^5}{2(1+t+t^2+t^3)}.$$

$$^d \phi_1 = \frac{7(1+t+t^2-t^3+3t^4)}{30(1+t+t^2+t^3+t^4)}, \phi_2 = \frac{2+2t+2t^2+2t^3-8t^4-5t^5+5t^6}{5(1+t+t^2+t^3+t^4)}.$$

Further computing the second factor on the right-hand side of (2.13) with the relation

$$\theta_2 - T = (1-t) \left\{ \theta_2 V_{m+1} - (m+2)t^{m-2} \left(\frac{\theta_1^2 t^2}{m} + \frac{H_0}{2} \right) \right\} \quad (2.14)$$

and simplified remaining terms in W , we favorably get the relation below:

$$g'''(\alpha) = \frac{6}{(m+1)(m+2)} (\phi_1 \theta_1^2 + \phi_2 \theta_2), \quad (2.15)$$

where

$$\begin{aligned} \phi_1 &= \frac{(m+2)}{m V_{m-1}} \left\{ V_m - t^{m-2} \left(t^2 + \frac{(m-1)(1-t)}{2(m+1)} \right) \right\} - \frac{m+2}{m+1}, \\ \phi_2 &= \frac{1}{V_{m-1}} \left\{ V_{m+1} + \frac{m+2}{m} V_{m-1} - 2t^{m-1} \right\} - 2. \end{aligned} \quad (2.16)$$

We summarize our analysis done so far in the following theorem:

Theorem 2.2. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ have a multiple zero α with integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α . Let θ_1 and θ_2 be defined as in Lemma 1.1. Let t be a root of $\rho(t)$ defined in (2.7). Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α . Then extended leap-frogging Newton's method (1.1) converges with order 3 and its asymptotic error constant η is given by

$$\eta = \frac{1}{(m+1)(m+2)} |\phi_1 \theta_1^2 + \phi_2 \theta_2|,$$

where ϕ_1 and ϕ_2 are described in (2.16), provided that $\phi_1 \theta_1^2 + \phi_2 \theta_2 \neq 0$.

Typical asymptotic error constants for $1 \leq m \leq 5$ are listed in Table 1 to confirm Theorem 2.2. To be more convenient, a unique real root t of $\rho(t)$ is chosen for an odd m , while any complex root t is chosen for an even m .

3. Algorithm, numerical results and discussions

The symbolic and computational ability of *Mathematica* [15] leads us to a zero-finding algorithm based on the analysis studied in Sections 1 and 2.

Algorithm 3.1 (Zero-Finding Algorithm).

- Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct iteration scheme (1.1) with the given function f at a multiple zero α as stated in Section 1.
 Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply η , p , θ_1 , θ_2 , t , ϕ_1 and ϕ_2 stated in Section 2. Set the error range ϵ , the maximum iteration number n_{\max} and the initial value x_0 . Compute $f(x_0)$ and $|x_0 - \alpha|$.
 Step 3. Compute x_{n+1} in (1.1) for $0 \leq n \leq n_{\max}$ and display the computed values of n , x_n , $|x_n - \alpha|$, $|e_{n+1}/e_n^p|$ and η .

Table 2Convergence behavior with $f(x) = (2 + x^6 + x^{10}) \sin(\pi x)$ and $(m, t, \alpha) = (1, 0, i)$.

n	x_n	$ x_n - \alpha $	e_{n+1}/e_n^3	η
0	0.900000000000000i	0.100000		47.65621492
1	0.918792522851259i	0.0812075	81.20747715	
2	0.954583408096213i	0.0454166	84.80598164	
3	0.993280916713669i	0.00671908	71.72431356	
4	0.999984580081552i	0.0000154199	50.83373301	
5	0.999999999999825i	1.74755×10^{-13}	47.66328781	
6	1.000000000000000i	2.54337×10^{-37}	47.65621492	
7	1.000000000000000i	7.84062×10^{-109}	47.65621492	
8	1.000000000000000i	$0. \times 10^{-250}$		

Table 3Convergence behavior with $f(x) = (-1 + e^{-30+7x+x^2})(x-3)^3$ and $(m, t, \alpha) = (4, 0.710616162493 + 0.657464589542i, 3)$.

n	x_n	$ x_n - \alpha $	e_{n+1}/e_n^3	η
0	2.900000000000000	0.100000		3.215825739
1	2.99880149920721 + 0.00235200343058868i	0.00263976	2.639758377	
2	$3.00000004037199 - (4.2349029 \times 10^{-8})i$	5.85093×10^{-8}	3.180770587	
3	$3.00000004037199 - (4.0228604 \times 10^{-23})i$	6.44121×10^{-22}	3.215826463	
4	$3.000000000000000 + (3.8439660 \times 10^{-64})i$	8.59398×10^{-64}	3.215825739	
5	$3.000000000000000 + (1.8183392 \times 10^{-189})i$	2.04115×10^{-189}	3.215825739	
6	$3.000000000000000 - (1.7433110 \times 10^{-261})i$	$0. \times 10^{-249}$		

Table 4Convergence behavior with $f(x) = (16 - 4x^2 + x^4)^2 \sin(\pi x/(\sqrt{3} + i))$ and $(m, t, \alpha) = (3, -0.421154814694, \sqrt{3} + i)$.

n	x_n	$ x_n - \alpha $	e_{n+1}/e_n^3	η
0	1.400000000000000 + 0.700000000000000i	0.447502		1.669881997
1	1.45935508781717 + 0.828398538435342i	0.322196	3.595304426	
2	1.60006531143506 + 1.01775900635815i	0.133175	3.981653644	
3	1.73626240826939 + 1.00292828779566i	0.00512957	2.171768467	
4	$1.73205102929788 + 1.00000000675362i$	2.21832×10^{-7}	1.643546296	
5	$1.73205080756888 + 1.00000000000000i$	1.82288×10^{-20}	1.669881146	
6	$1.73205080756888 + 1.00000000000000i$	1.01148×10^{-59}	1.669881997	
7	$1.73205080756888 + 1.00000000000000i$	1.72806×10^{-177}	1.669881997	
8	$1.73205080756888 + 1.00000000000000i$	$0. \times 10^{-249}$		

In these experiments, to achieve the specified sufficient accuracy, we choose 250 as the minimum number of digits of precision by assigning \$MinPrecision=250 in Mathematica Version7. We set the error bound ϵ to 0.5×10^{-235} for $|x_n - \alpha| < \epsilon$ and evaluate the n th order derivative of the complicated nonlinear functions using the Mathematica command **D[f, {x, n}]**. Although computed values of x_n and α are rounded to be accurate up to 235 significant digits, the limited space allows us to list them only up to 15 significant digits.

The cubic convergence of the proposed numerical method is shown to be successful by selecting t , a root of $\rho(t)$ introduced in (2.7). Although any complex or real root t can give us the cubic convergence, we conveniently select t based on Theorem 2.1. It is chosen as a unique real root t when m is odd, while as any complex root t when m is even.

As a first example, we illustrate the order of convergence and the asymptotic error constant with a function

$$f(x) = (2 + x^6 + x^{10}) \sin(\pi x)$$

having a simple complex zero $\alpha = i = \sqrt{-1}$ with $t = 0$. We choose $x_0 = 0.9i$ as an initial guess. Table 2 shows a good agreement with the theory developed in this paper.

As a second example for the convergence of extended leap-frogging Newton's method, we take

$$f(x) = (-1 + e^{-30+7x+x^2})(x-3)^3$$

with a zero 3 of multiplicity 4 with $t \simeq 0.710616162493329 + 0.6574645895427713i$. We choose $x_0 = 2.9$ as an initial guess. We confirm that the order of convergence is cubic and the computed asymptotic error constant agrees with the theoretical value. Table 3 lists the numerical results for approximated zeros of $f(x)$ computed with Mathematica programming.

As another numerical example to confirm the convergence, we take

$$f(x) = (16 - 4x^2 + x^4)^2 \sin(\pi x/(\sqrt{3} + i))$$

with a zero $\sqrt{3} + i$ of multiplicity 3. The results for this example are displayed in Table 4 clearly reflecting the theoretical convergence presented in this paper.

Table 5

Convergence behavior for various test functions.

f	α	m	t	x_0	$ x_n - \alpha $	ν	η
f_1	0.73908513321516	1	0	0.490	$0. \times 10^{-250}$	5	0.048755022
f_2	1.4044916482153	2	$0.25 + i\sqrt{7}/4$	1.290	$0. \times 10^{-249}$	6	0.986524964
f_3	$\sqrt{2}$	3	-0.4211548146	1.080	$0. \times 10^{-249}$	7	146.1859725
f_4	2	4	$0.71061 + 0.65746i$	1.95	$0. \times 10^{-249}$	5	0.166204568
f_5	$3 + i\sqrt{5}$	5	-0.544435	$2.5 + 2.5i$	$0. \times 10^{-249}$	6	0.174392126
f_6	$1 - i\sqrt{\pi}$	6	$-0.50627 - 0.30937i$	$1.3 - 1.7i$	0×10^{-249}	8	15.98966748
f_7	π	7	-0.6192465989	2.590	$0. \times 10^{-249}$	7	154.9862056
f_8	2	8	$-0.59809 - 0.25560i$	1.590	1.4×10^{-261}	6	0.074985694

We further confirm our analysis through more test functions that are listed below:

$$f_1(x) = \cos x - x, \quad \alpha = 0.739085133215161, \quad m = 1.$$

$$f_2(x) = (\sin^2 x - x^2 + 1)(\cos 2x + 2x^2 - 3), \quad \alpha = 1.40449164821534, \quad m = 2$$

$$f_3(x) = (x^2 - 2)^2(\sin(\pi x/2\sqrt{2}) - x^4 + 3), \quad \alpha = \sqrt{2}, \quad m = 3.$$

$$f_4(x) = (x \sin(\pi x/4) - 2)(x - 2)^2 \log(x - 1), \quad \alpha = 2, \quad m = 4.$$

$$f_5(x) = (14 - 6x + x^2)^3(1 + \cos(\pi x/(3 + i\sqrt{5}))), \quad \alpha = 3 + i\sqrt{5}, \quad m = 5.$$

$$f_6(x) = (x^2 - 2x + 1 + \pi)^3(\log[x^2 - 2x + 2 + \pi])^3, \quad \alpha = 1 - i\sqrt{\pi}, \quad m = 6.$$

$$f_7(x) = (e^{-x} \sin x + \log[1 + (x - \pi)^2])(x - \pi) \sin^3 x (\log(x - \pi + 1))^2, \quad \alpha = \pi, \quad m = 7.$$

$$f_8(x) = (x^2 \sin(\pi x/8) + e^{(x-2)^2} - 1 - 2\sqrt{2})(x - 2)^3 \sin^4(\pi x/2), \quad \alpha = 2, \quad m = 8.$$

Table 5 shows a successful convergence behavior within a given error bound $\epsilon = \frac{1}{2} \times 10^{-235}$ for the above test functions with the multiplicity m , $t = 1 - \mu/m \neq 1$ such that $\rho(t) = 0$, the initial guess x_0 , the least iteration number ν and the asymptotic error constant η . When $m \geq 5$, the root t of $\rho(t)$ can be found numerically. Although not listed in Tables 2–5, the residual error of $f(x)$ for each step is found to be sufficiently near zero indicating a very accurate numerical solution.

Various numerical experiments have verified the cubic order of convergence and the asymptotic error constant of the proposed leap-frogging Newton's method. With a properly chosen t namely μ for a given m , it is worthwhile to derive a high-order numerical method that can locate multiple zeros. The theoretical development suggested in this paper will play a significant role in establishing a numerical method that accurately finds approximate zeros of a nonlinear algebraic equation. The current study can be easily applied to a different numerical method locating a multiple zero with high accuracy.

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